ON SIMPLE FILIPPOV SUPERALGEBRAS OF TYPE $A(m, n)$

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Abstract: It is proved that there exist no simple finite-dimensional Filippov superalgebras of type $A(m, n)$ over an algebraically closed field of characteristic 0.

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1 Introduction

The concept of n-Lie superalgebra was presented by Daletskii and Kushnirevich, in [3], as a natural generalization of the n-Lie algebra notion introduced by Filippov in 1985 (see [4]). Following [5] and [10], in this article, we use the terms Filippov superalgebra and Filippov algebra instead of n-Lie superalgebra and n-Lie algebra, respectively. Filippov algebras were also known before under the names of Nambu Lie algebras and Nambu algebras. As pointed out in [14] and [15], Filippov algebras are a particular case of n-ary Malcev algebras (generalizing the fact that every Lie algebra is a Malcev algebra). We may also remark that a 2-Lie superalgebra is simply known as a Lie superalgebra. The description of the finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero was given by Kac in [7].

This work is one more step on the way to the classification of finite-dimensional simple Filippov superalgebras over an algebraically closed field of characteristic 0. In [9], finite-dimensional commutative n-ary Leibniz algebras over a field of characteristic 0 were studied by the second author. He showed that there exist no simple ones. The finite-dimensional simple Filippov algebras over an algebraically closed field of characteristic 0 were classified by Ling in [8]. Notice that an n-ary commutative Leibniz algebra is exactly a Filippov superalgebra with trivial even part, and a Filippov algebra is exactly a Filippov superalgebra with trivial odd part. Bearing in mind these facts, we consider the n-ary Filippov superalgebras with $n \geq 3$, and with nonzero even and odd parts.

Let G be a Lie superalgebra. We say that a Filippov superalgebra $\mathcal{F}$ has *type* G if $Inder(\mathcal{F}) \cong G$ (see definitions below). A description of simple Filippov superalgebras of type $B(m, n)$ was already obtained in [11], [13] and [12]. The same problem concerning

Filippov superalgebras of type $A(m,n)$ with $m = n$ has recently been solved in [1]. Moreover, the type $A(0,n)$, with $n \in \mathbb{N}$, was studied in [2]. The present work represents the final step towards the classification of finite-dimensional simple Filippov superalgebras of type $A(m,n)$ over an algebraically closed field of characteristic zero. Concretely, we establish a negative answer to the existence problem of the mentioned superalgebras when $m, n \in \mathbb{N}$ and $m \neq n$.

We start recalling some definitions.

An Ω-*algebra* over a field k is a linear space over k equipped with a system of multilinear algebraic operations $\Omega = \{\omega_i : |\omega_i| = n_i \in \mathbb{N},\ i \in I\}$, where $|\omega_i|$ denotes the arity of ω_i.

An n-*ary Leibniz algebra* over a field k is an Ω-algebra L over k with one n-ary operation $(\cdot, \cdots, \cdot)$ satisfying the identity

$$((x_1, \ldots, x_n), y_2, \ldots, y_n) = \sum_{i=1}^{n} (x_1, \ldots, (x_i, y_2, \ldots, y_n), \ldots, x_n).$$

If this operation is anticommutative, we obtain the definition of *Filippov* (n-*Lie*) algebra over a field.

An n-*ary superalgebra* over a field k is a $\mathbb{Z}_2$-graded n-ary algebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ over k, that is,

$$\text{if } x_i \in L_{\alpha_i}, \alpha_i \in \mathbb{Z}_2, \text{ then } (x_1, \ldots, x_n) \in L_{\alpha_1 + \ldots + \alpha_n}.$$

An n-*ary Filippov superalgebra* over k is an n-ary superalgebra $\mathcal{F} = \mathcal{F}_{\bar{0}} \oplus \mathcal{F}_{\bar{1}}$ over k with one n-ary operation $[\cdot, \cdots, \cdot]$ satisfying

$$[x_1, \ldots, x_{i-1}, x_i, \ldots, x_n] = -(-1)^{p(x_{i-1})p(x_i)}[x_1, \ldots, x_i, x_{i-1}, \ldots, x_n], \tag{1}$$

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} (-1)^{p\bar{q}_i}[x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n], \tag{2}$$

where $p(x) = l$ means that $x \in \mathcal{F}_{\bar{l}}$, $p = \sum_{i=2}^{n} p(y_i)$, $\bar{q}_i = \sum_{j=i+1}^{n} p(x_j)$, $\bar{q}_n = 0$. The identities (1) and (2) are called the anticommutativity and the generalized Jacobi identity, respectively. By (1), we can rewrite (2) as

$$[y_2, \ldots, y_n, [x_1, \ldots, x_n]] = \sum_{i=1}^{n} (-1)^{pq_i}[x_1, \ldots, [y_2, \ldots, y_n, x_i], \ldots, x_n], \tag{3}$$

where $q_i = \sum_{j=1}^{i-1} p(x_j)$, $q_1 = 0$. Sometimes, instead of using the long term "n-ary superalgebra", we simply say for short "superalgebra". If we denote by $L_x = L_{(x_1, \ldots, x_{n-1})}$ the operator of left multiplication $L_x y = [x_1, \ldots, x_{n-1}, y]$, then, by (3), we get

$$[L_y, L_x] = \sum_{i=1}^{n-1} (-1)^{pq_i} L(x_1, \ldots, L_y x_i, \ldots, x_{n-1}),$$

where L_y is an operator of left multiplication and p its parity. (Here and afterwards, we denote the supercommutator by $[\,,]$).

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be an n-ary anticommutative superalgebra. A *subsuperalgebra* $B = B_{\bar{0}} \oplus B_{\bar{1}}$ of the superalgebra L, $B_{\bar{i}} \subseteq L_{\bar{i}}$, is a $\mathbb{Z}_2$-graded vector subspace of L such that $[B, \ldots, B] \subseteq B$. A subsuperalgebra I of L is called an *ideal* if $[I, L, \ldots, L] \subseteq I$. The subalgebra (in fact, an ideal) $L^{(1)} = [L, \ldots, L]$ of L is called the *derived subsuperalgebra* of L. Put $L^{(i)} = [L^{(i-1)}, \ldots, L^{(i-1)}]$, $i \in \mathbb{N}, i > 1$. The superalgebra L is called *solvable* if $L^{(k)} = 0$ for some k. Denote by $R(L)$ the maximal solvable ideal of L (if it exists). If $R(L) = 0$, the superalgebra L is called *semisimple*. The superalgebra L is called *simple* if $L^{(1)} \neq 0$ and L lacks ideals other than 0 or L.

The article is organized as follows.

In the second section we recall how to reduce the classification problem of simple Filippov superalgebras to some question about Lie superalgebras, using the same ideas as in [8]. Concretely, we consider an existence problem for some skewsymmetric homomorphisms of semisimple Lie superalgebras and their faithful irreducible modules. This section is followed with the third one where we collect some definitions and results on Lie superalgebras that we will apply in the two last sections. We also fix some notations with the same purpose.

The fourth section is devoted to the problem of existence of finite-dimensional simple Filippov superalgebras of type $A(m, n)$ with $m \neq n$. We start with the particular case $A(1, n)$ in the first subsection, where, taking into account [1] and [2], it is assumed that $n \in \mathbb{N} \setminus \{1\}$. The main result of this article (Theorem 4.2) is stated and proved in the second subsection.

In each of the two mentioned subsections we restrict our considerations to the case of the Lie superalgebra that gives the name to the type and solve the existence problem of the mentioned skewsymmetric homomorphisms. It turns out that the required homomorphisms do not exist. Therefore, there are no simple finite-dimensional Filippov superalgebras of type $A(m, n)$ over an algebraically closed field of characteristic 0. Moreover, as a corollary of its proof, we see that there is no simple finite-dimensional Filippov superalgebra $\mathcal{F}$ of type $A(m, n)$ such that $\mathcal{F}$ is a highest weight module over $A(m, n)$.

In what follows, by Φ we denote an algebraically closed field of characteristic 0, by F a field of characteristic 0, by k a field and by $\langle w_\upsilon;\ \upsilon \in \Upsilon \rangle$ a linear space spanned by the family of vectors $\{w_\upsilon;\ \upsilon \in \Upsilon\}$ over a field (the field is clear from the context). The symbol := denotes an equality by definition.

2 Reduction to Lie superalgebras

From now on, we denote by $\mathcal{F}$ an n-ary Filippov superalgebra. Let us denote by $\mathcal{F}^*$ ($L(\mathcal{F})$) the associative (Lie) superalgebra generated by the operators $L(x_1, \ldots, x_{n-1})$, $x_i \in \mathcal{F}$. The algebra $L(\mathcal{F})$ is called *the algebra of multiplications* of $\mathcal{F}$.

Lemma 2.1 [11] *Let $\mathcal{F} = \mathcal{F}_{\bar{0}} \oplus \mathcal{F}_{\bar{1}}$ be a simple finite-dimensional Filippov superalgebra over a field of characteristic* 0 *with $\mathcal{F}_{\bar{1}} \neq 0$. Then $L = L(\mathcal{F}) = L_{\bar{0}} \oplus L_{\bar{1}}$ has nontrivial even and odd parts.*

Theorem 2.1 [11] *If $\mathcal{F}$ is a simple finite-dimensional Filippov superalgebra over a field of characteristic* 0*, then $L = L(\mathcal{F})$ is a semisimple Lie superalgebra.*

Given an n-ary superalgebra A with a multiplication $(\cdot, \cdots, \cdot)$, we have $End(A) = End_{\bar{0}} A \oplus End_{\bar{1}} A$. The element $D \in End_{\bar{s}} A$ is called a *derivation* of degree s of A if, for every $a_1, \ldots, a_n \in A, p(a_i) = p_i$, the following equality holds

$$D(a_1, \ldots, a_n) = \sum_{i=1}^{n} (-1)^{sq_i} (a_1, \ldots, Da_i, \ldots, a_n),$$

where $q_i = \sum_{j=1}^{i-1} p_j$. We denote by $Der_{\bar{s}} A \subset End_{\bar{s}} A$ the subspace of all derivations of degree s and set $Der(A) = Der_{\bar{0}} A \oplus Der_{\bar{1}} A$. The subspace $Der(A) \subset End(A)$ is easily seen to be closed under the bracket

$$[a, b] = ab - (-1)^{deg(a)deg(b)} ba$$

(known as the *supercommutator*) and it is called *the superalgebra of derivations* of A.

Fix $n-1$ elements $x_1, \ldots, x_{n-1} \in A$, $i \in \{1, \ldots, n\}$, and define a transformation $ad_i(x_1, \ldots, x_{n-1}) \in End(A)$ by the rule

$$ad_i(x_1, \ldots, x_{n-1})\, x = (-1)^{pq_i} (x_1, \ldots, x_{i-1}, x, x_i, \ldots, x_{n-1}),$$

where $p = p(x), p_i = p(x_i), q_i = \sum_{j=i}^{n-1} p_j$.

If, for all $i = 1, \ldots, n$ and $x_1, \ldots, x_{n-1} \in A$, the transformations $ad_i(x_1, \ldots, x_{n-1}) \in End(A)$ are derivations of A, then we call them *strictly inner derivations* and A *an inner-derivation superalgebra* ($\mathcal{ID}$*-superalgebra*). Notice that the n-ary Filippov superalgebras and the n-ary commutative Leibniz algebras are examples of $\mathcal{ID}$-superalgebras.

Now let us denote by $Inder(A)$ the linear space spanned by the strictly inner derivations of A. If A is an n-ary $\mathcal{ID}$-superalgebra then it is easy to see that $Inder(A)$ is an ideal of $Der(A)$.

Lemma 2.2 [11] *Given a simple $\mathcal{ID}$-superalgebra A over k, the Lie superalgebra $Inder(A)$ acts faithfully and irreducibly on A.*

Let $\mathcal{F}$ be an n-ary Filippov superalgebra over k. Notice that the map $ad := ad_n : \otimes^{n-1}\mathcal{F} \mapsto Inder(\mathcal{F})$ satisfies

$$[D, ad(x_1, \ldots, x_{n-1})] = \sum_{i=1}^{n-1} (-1)^{pq_i} ad(x_1, \ldots, x_{i-1}, Dx_i, x_{i+1}, \ldots, x_{n-1}),$$

for all $D \in Inder(\mathcal{F})$, and the associated map $(x_1, \ldots, x_n) \mapsto ad(x_1, \ldots, x_{n-1})\, x_n$ from $\otimes^n \mathcal{F}$ to $\mathcal{F}$ is $\mathbb{Z}_2$-skewsymmetric. If we regard $\mathcal{F}$ as an $Inder(\mathcal{F})$-module then ad induces an $Inder(\mathcal{F})$-module morphism from the $(n-1)$-th exterior power $\wedge^{n-1}\mathcal{F}$ to $Inder(\mathcal{F})$ (which we also denote by ad) such that the map $(x_1, \ldots, x_n) \mapsto ad(x_1, \ldots, x_{n-1})\, x_n$ is $\mathbb{Z}_2$-skewsymmetric. (Note that in $\wedge^{n-1}\mathcal{F}$ we have $x_1 \wedge \ldots \wedge x_i \wedge x_{i+1} \wedge \ldots \wedge x_{n-1} = -(-1)^{p_i p_{i+1}} x_1 \wedge \ldots \wedge x_{i+1} \wedge x_i \wedge \ldots \wedge x_{n-1}$.) Conversely, if (L, V, ad) is a triple with L a Lie superalgebra, V an L-module, and ad an L-module morphism from $\wedge^{n-1}V \mapsto L$ such that the map $(v_1, \ldots, v_n) \mapsto ad(v_1 \wedge \ldots \wedge v_{n-1})\, v_n$ from $\otimes^n V$ to V is $\mathbb{Z}_2$-skewsymmetric (we call the homomorphisms of this type *skewsymmetric*), then V becomes an n-ary Filippov superalgebra by defining

$$[v_1, \ldots, v_n] = ad(v_1 \wedge \ldots \wedge v_{n-1})\, v_n.$$

Therefore, we obtain a correspondence between the set of n-ary Filippov superalgebras and the set of triples (L, V, ad), satisfying the conditions above.

We assume that all vector spaces appearing in the following are finite-dimensional over F.

If $\mathcal{F}$ is a simple n-ary Filippov superalgebra then Theorem 2.1 shows that the Lie superalgebra $Inder(\mathcal{F})$ is semisimple, and $\mathcal{F}$ is a faithful and irreducible $Inder(\mathcal{F})$-module. Moreover, the $Inder(\mathcal{F})$-module morphism $ad : \wedge^{n-1}\mathcal{F} \mapsto Inder(\mathcal{F})$ is surjective.

Conversely, if (L, V, ad) is a triple such that L is a semisimple Lie superalgebra over F, V is a faithful irreducible L-module, ad is a surjective L-module morphism from $\wedge^{n-1}V$ onto the adjoint module L, and the map $(v_1, \ldots, v_n) \mapsto ad(v_1 \wedge \ldots \wedge v_{n-1})\, v_n$ from $\otimes^n V$ to V is $\mathbb{Z}_2$-skewsymmetric, then the corresponding n-ary Filippov superalgebra is simple. A triple with these conditions will be called a *good triple.* Thus, the problem of determining the simple n-ary Filippov superalgebras over F can be translated to that of finding the good triples.

3 Some notations and results on Lie superalgebras

In this section, we recall some notations and results from [7] on the Lie superalgebra $A(m, n)$ (and its irreducible faithful finite-dimensional representations). We also give some explicit constructions which we shall use some later in the study of the simple finite-dimensional Filippov superalgebras of type $A(m, n)$. Let us start recalling the definition of induced module.

Let $\mathcal{L}$ be a Lie superalgebra, $U(\mathcal{L})$ its universal enveloping superalgebra [7], H a subalgebra of $\mathcal{L}$, and V an H-module. The module V can be extended to $U(H)$-module. We consider the $\mathbb{Z}_2$-graded space $U(\mathcal{L}) \otimes_{U(H)} V$ (this is the quotient space of $U(\mathcal{L}) \otimes V$ by the linear span of the elements of the form $gh \otimes v - g \otimes h(v)$, $g \in U(\mathcal{L})$, $h \in U(H)$). This space can be endowed with the structure of a $\mathcal{L}$-module as follows $g(u \otimes v) = gu \otimes v, g \in \mathcal{L}, u \in U(\mathcal{L}), v \in V$. The so-constructed $\mathcal{L}$-module is said to be *induced from the H-module V* and is denoted by $Ind_H^{\mathcal{L}} V$.

From now on, we denote by G a contragredient Lie superalgebra over Φ and consider it with the "standard" $\mathbb{Z}$-grading [7, Sections 5.2.3 and 2.5.7].

Let $G = \oplus_{i \geq -d} G_i$. Set $H = (G_0)_{\bar{0}} = \langle h_1, \ldots, h_n \rangle$, $N^+ = \oplus_{i>0} G_i$ and $B = H \oplus N^+$. Let $\Lambda \in H^*, \Lambda(h_i) = a_i \in \Phi$, $\langle v_\Lambda \rangle$ be an one-dimensional B-module for which $N^+(v_\Lambda) = 0, h_i(v_\Lambda) = a_i v_\Lambda$. Let $\delta_i \in H^*$, $\delta_i(h_j) = \delta_{ij}$ where δ_{ij} is Kronecker's delta. Let $V_\Lambda = Ind_B^G \langle v_\Lambda \rangle / I_\Lambda$, where I_Λ is the (unique) maximal submodule of the G-module V_Λ. Then Λ is called the *highest weight* of the G-module V_Λ. By [7], every faithful irreducible finite-dimensional G-module may be obtained in this manner. Note that the condition $1 \otimes v_\Lambda \in V_{\bar{0}}(V_{\bar{1}})$ gives a $\mathbb{Z}_2$-grading on V_Λ.

Lemma 3.1 [11] *Let V be a module over a Lie superalgebra G, let $V = \oplus V_{\gamma_i}$ be its weight decomposition, and let ϕ be a homomorphism from $\wedge^m V$ into G. Then, for all $v_i \in V_{\gamma_i}$,*

$$\begin{array}{ll} \phi(v_1, \ldots, v_m) \in G_{\gamma_1 + \ldots + \gamma_m}, & \text{if } \gamma_1 + \ldots + \gamma_m \text{ is a root of } G, \\ \phi(v_1, \ldots, v_m) = 0, & \text{otherwise.} \end{array}$$

Let G be a contragredient Lie superalgebra of rank n, $U = Ind_B^G \langle v_\Lambda \rangle$, and $V = V_\Lambda = U/N$ be a finite-dimensional representation of G, where $N = I_\Lambda$ is a maximal proper submodule of the G-module V_Λ. Let $G = \oplus_\alpha G_\alpha$ be a root decomposition of G relative to a Cartan subalgebra H. Denote by $\mathcal{A}$ the following set of roots: $\mathcal{A} = \{\alpha : g_\alpha \notin B\}$.

Lemma 3.2 [13] *Let $g_\alpha \in G_\alpha$ and $g_\alpha \otimes v \neq 0$ ($v = v_\Lambda$). Then*

$$g_\alpha^j \otimes v \in U_{\sum_{i=1}^n (j\alpha(h_i)+\Lambda(h_i))\delta_i}$$

for all $j \in \mathbb{N}$, and there exists a minimal $k \in \mathbb{N}$ such that $g_\alpha^k \otimes v \in N$. Moreover, the set $\mathcal{E}_{\alpha,k} = \{1 \otimes v, g_\alpha \otimes v, \ldots, g_\alpha^{k-1} \otimes v\}$ is linearly independent in V. Setting $h = [g_{-\alpha}, g_\alpha]$, we have

1. *$\Lambda(h) = -\frac{(k-1)\alpha(h)}{2}$ if either $g_\alpha \in G_{\bar{0}}$ or $k \notin 2\mathbb{N}$;*
2. *$\alpha(h) = 0$ if $g_\alpha \in G_{\bar{1}}$ and $k \in 2\mathbb{N}$.*

Remark 3.1 *Note that if we start with a root β then there exists $s \in \mathbb{N}$ such that $\mathcal{E}_{\beta,s}$ is linearly independent, but $\mathcal{E}_{\alpha,k} \cup \mathcal{E}_{\beta,s}$ may not be linearly independent.*

Recall that a set $\mathcal{E}$ is called a *pre-basis* of a vector space W if $\langle \mathcal{E} \rangle = W$.

Let $\{g_{\alpha_1}^{k_1} \ldots g_{\alpha_s}^{k_s} \otimes v; k_i \in \mathbb{N}_0, \alpha_i \in \mathcal{A}\}$ be a pre-basis of U. As we have seen above, for every $i = 1, \ldots, s$, there exists a minimal $p_i \in \mathbb{N}$ such that $g_{\alpha_i}^{p_i} \otimes v \in N$. Using the induction on the word length, it is easy to show that $\{g_{\alpha_1}^{k_1} \ldots g_{\alpha_s}^{k_s} \otimes v; k_i \in \mathbb{N}_0, k_i < p_i, \alpha_i \in \mathcal{A}\}$ is a pre-basis of U/N.

We finish this part with some more notations that we use in the two next sections:
- the symbol $\doteq$ denotes an equality up to a nonzero coefficient;
- $\underline{u, v}_{\,t}$ means that the elements u and v are t-times repeating $\underbrace{u, v, \ldots, u, v}_{2t}$, being the index t omitted when its value is clear from the context.

4 Simple Filippov superalgebras of type $A(m, n)$

In what follows, considering $A(m,n)$, we assume that $m \neq n$. Recall that $A(m,n) := sl(m+1, n+1)$ for $m \neq n$ and $m, n \in \mathbb{N}_0$. It consists of the matrices of type

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right),$$

where $A \in M_{(m+1)\times(m+1)}(F), B \in M_{(m+1)\times(n+1)}(F), C \in M_{(n+1)\times(m+1)}(F), D \in M_{(n+1)\times(n+1)}(F)$ and $tr(A) = tr(D)$. Let us write some elements in $G = A(m,n)$:

$$\left.\begin{array}{l} h_i = \ e_{ii} - e_{i+1,i+1}, \quad i = 1, \ldots, m, m+2, \ldots, m+n+1, \\ h_{m+1} = e_{m+1,m+1} + e_{m+2,m+2}, \\ e_{kl} := g_{\epsilon_k - \epsilon_l} \in G_{\epsilon_k - \epsilon_l}, \quad k, l = 1, \ldots, m+1 \text{ or } k, l = m+2, \ldots, m+n+2, \end{array}\right\} \in G_{\bar{0}}$$

$$\left.\begin{array}{r} e_{kl} := g_{\epsilon_k - \epsilon_l} \in G_{\epsilon_k - \epsilon_l}, \quad k = 1, \ldots, m+1, l = m+2, \ldots, m+n+2, \\ \text{or } k = m+2, \ldots, m+n+2, l = 1, \ldots, m+1. \end{array}\right\} \in G_{\bar{1}}$$

The space $H := G_0 = \langle h_1, \ldots, h_{m+n+1} \rangle$ is a Cartan subalgebra of $A(m,n)$, and ϵ_i are the linear functions on H defined by its values on $h_1, \ldots, h_{m+n+1}$ and the conditions $\epsilon_i(e_{jj}) = \delta_{ij}$, where δ_{ij} is Kronecker's delta. Then $\Delta = \Delta_0 \cup \Delta_1$ is a root system for $A(m,n)$, where $\Delta_0 = \{0; \epsilon_k - \epsilon_l, k, l = 1, \ldots, m+1 \text{ or } k, l = m+2, \ldots, m+n+2\}$, and $\Delta_1 = \{\epsilon_k - \epsilon_l, k = 1, \ldots, m+1, l = m+2, \ldots, m+n+2 \text{ or } k = m+2, \ldots, m+n+2, l = 1, \ldots, m+1\}$. The roots $\{\alpha_i := \epsilon_i - \epsilon_{i+1}, i = 1, \ldots, m+n+1\}$ are simple.

The conditions $deg\ g_{\alpha_i} = 1, deg\ g_{-\alpha_i} = -1$ give us the standard grading of $A(m,n)$, [7, Section 5.2.3]. The negative part of this grading is $G_{\epsilon_k - \epsilon_l}, l < k$. Because of this, the set

$$\mathcal{E} = \left\{ \prod_{l<k} g_{\epsilon_k - \epsilon_l}^{\gamma_{kl}} \otimes v \ : \ \gamma_{kl} \in \mathbb{N}_0 \right\} \tag{4}$$

is a pre-basis of the induced module $M = Ind_B^G \langle v_\Lambda \rangle$ (where $v = v_\Lambda$).

Let V be an irreducible module over $G = A(m,n)$ with the highest weight Λ, $\Lambda(h_i) = a_i$. Denote Λ by $(a_1, \ldots, a_{m+n+1})$. Applying Lemma 3.2, we have

$$a_i \in \mathbb{N}_0 \ \text{ if } i \neq m+1. \tag{5}$$

If $u \in V_\gamma$ (or G_γ) then we may write $h_q u = p_q(u)u$ ($[h_q, u] = p_q(u)u$) and call *q-weight* of u to $p_q(u)$.

In what follows, the symbol $w \overset{i}{\underset{j}{\otimes}} v$ means that $p_{h_l}(w \otimes v) = p_l(w \otimes v) = i$ and $p_{h_r}(w \otimes v) = p_r(w \otimes v) = j$ (the same with the notation $\overset{i}{\underset{j}{u}}$), where everytime in text we specify the indices l and r.

4.1 The type $A(1,n)$

In this subsection, because of [1] and [2], we assume that $n \in \mathbb{N} \setminus \{1\}$. We begin with some technical lemmas on irreducible modules of some special types over $A(1,n)$.

Lemma 4.1 *Let $V = V_\Lambda$ be an irreducible module over $G = A(1,n)$ with $\Lambda = (a_1, \ldots, a_{n+2})$, $a_1 = 1$ and $a_2 \neq 0$. Assume that (G, V, ϕ) is a good triple. Then $a_2 \leq -\frac{1}{2}$.*

Proof. Suppose that $\phi(u_1, \ldots, u_s) = g_{\epsilon_3 - \epsilon_2}$. Consider $h = h_1 + h_2 = e_{11} + e_{33}$. From the nonzero action on $1 \otimes v$, we obtain $|2 - p_h(u_i) + a_2| \leq 1$. If $a_2 > -\frac{1}{2}$ then $p_h(u_i) > \frac{1}{2}$, which leads to a contradiction since $p_h(g_{\epsilon_3 - \epsilon_2}) = 1$. Thus, $a_2 \leq -\frac{1}{2}$. ∎

Lemma 4.2 *Let $V = V_\Lambda$ be an irreducible module over $A(1,n)$ with $\Lambda = (a_1, \ldots, a_{n+2})$, $a_1 = 1$ and $a_2 = \ldots = a_{2+n} = 0$. Then*

$$\left\{ \prod_{i=2}^{3+n} g_{\epsilon_i - \epsilon_1}^{\alpha_i} \otimes v : \alpha_i \in \{0, 1\} \right\}$$

is a pre-basis of V.

Proof. Consider $g_{\epsilon_i-\epsilon_1}$ with $i \neq 1$. Suppose that $g_{\epsilon_i-\epsilon_1} \in G_{\bar{0}}$. Then, by Lemma 3.2, $g^2_{\epsilon_i-\epsilon_1} \otimes v = 0$. If $g_{\epsilon_i-\epsilon_1} \in G_{\bar{1}}$ then, from $[g_{\epsilon_i-\epsilon_1}, g_{\epsilon_i-\epsilon_1}] \otimes v = 0$, we obtain the same conclusion. ■

Lemma 4.3 *Let $V = V_\Lambda$ be an irreducible module over $A(1,n)$ with $\Lambda = (a_1, \dots, a_{n+2})$, $a_2 = a \neq 0$ and $a_i = 0$ for $i \neq 2$. Suppose that $h = e_{11} + e_{33}$ and $\phi(u_1, \dots, u_s) = g_{\epsilon_3-\epsilon_2}$. Then it is impossible that $p_h(u_i) = a$ for all $i \in \{1, \dots, s\}$.*

Proof. Notice that $w_0 = \phi(1 \otimes v, u_2, \dots, u_s) \in \mathcal{E}_1$, where $\mathcal{E}_1$ denotes the elements from $A(1,n)$ with h-weight equal to 1: $\mathcal{E}_1 = \{g_{\epsilon_1-\epsilon_i}, g_{\epsilon_3-\epsilon_i} : i \neq 1, 3\}$. If $w_0 = g_{\epsilon_3-\epsilon_i}$ then we can multiply w_0 by $g_{\epsilon_i-\epsilon_{n+3}}$ to think that $w_0 = g_{\epsilon_3-\epsilon_{n+3}}$. We may proceed analogously with $g_{\epsilon_1-\epsilon_i}$. Now, if $w_0 = g_{\epsilon_1-\epsilon_{n+3}}$ then we may multiply it by $g_{\epsilon_3-\epsilon_1}$ to arrive at $g_{\epsilon_3-\epsilon_{n+3}}$. Thus, we may replace all u_i either with $g_{\epsilon_{j_i}-\epsilon_2} \otimes v$ ($j_i > 3$) or $1 \otimes v$ (maybe, multiplied by $\alpha := g_{\epsilon_3-\epsilon_1}$). Thus, we arrive at

$$w = \phi(\alpha^{\delta_1} g_{\epsilon_{i_1}-\epsilon_2} \otimes v, \dots, \alpha^{\delta_r} g_{\epsilon_{i_r}-\epsilon_2} \otimes v, \alpha^{\delta_{r+1}} g_{\epsilon_{n+3}-\epsilon_2} \otimes v, \dots,$$

$$\dots, \alpha^{\delta_{r+q}} g_{\epsilon_{n+3}-\epsilon_2} \otimes v, \underline{\alpha \otimes v}_t, \underline{1 \otimes v}_p) \doteq g_{\epsilon_3-\epsilon_{n+3}},$$

where $3 < i_j < n+3, \delta_k \in \{0,1\}$. The action of h on w gives $a(r+q+t+p) = 1$, and the action of $e_{22} + e_{n+3,n+3}$ gives $r = 2$. The action of $e_{n+2,n+2} - e_{n+3,n+3}$ gives either $q = 1$ or $q = 0$.

If $\mathbf{q = 1}$ then $i_1 = i_2 = n+2$, and the action of $e_{33} - e_{n+2,n+2}$ implies $t + \sum_{i=1}^3 \delta_i = 3$. If $\delta_3 = 0$ then $g_{\epsilon_{n+3}-\epsilon_2} \otimes v \curvearrowright g_{\epsilon_{n+3}-\epsilon_1} \otimes v$ gives an element with $(e_{11} + e_{n+3,n+3})$-weight being equal to -2. If $\delta_1 = 0$ (or $\delta_2 = 0$) then $g_{\epsilon_{n+2}-\epsilon_2} \otimes v \curvearrowright g_{\epsilon_{n+3}-\epsilon_1} \otimes v$ leads to an element with $(e_{11} + e_{n+2,n+2})$-weight being equal to -2. Thus, $\delta_1 = \delta_2 = \delta_3 = 1, t = 0$. Consider the action of w on $g_{\epsilon_{n+3}-\epsilon_1} \otimes v$ with the consecutive change $\alpha g_{\epsilon_{n+3}-\epsilon_2} \otimes v \curvearrowright g_{\epsilon_{n+3}-\epsilon_1} \otimes v$. We get

$$w_1 = \phi(\alpha g_{\epsilon_{n+2}-\epsilon_2} \otimes v, \alpha g_{\epsilon_{n+2}-\epsilon_2} \otimes v, g_{\epsilon_{n+3}-\epsilon_1} \otimes v, \underline{1 \otimes v}_p),$$

and $p_1(w_1) = -1, p_2(w_1) = p_{n+2}(w_1) = 1$, i.e., $w_1 \doteq g_{\epsilon_2-\epsilon_{n+3}}$. The action of w_1 on $g_{\epsilon_{n+3}-\epsilon_2} \otimes v$ gives

$$w_2 = \phi(\alpha g_{\epsilon_{n+2}-\epsilon_2} \otimes v, g_{\epsilon_{n+3}-\epsilon_2} \otimes v, g_{\epsilon_{n+3}-\epsilon_1} \otimes v, \underline{1 \otimes v}_p) \neq 0$$

(note that we may assume $n = 2$, since otherwise the $(e_{22} + e_{44})$-weight of w is -2). Then $p_1(w_2) = p_2(w_2) = p_3(w_2) = 0, p_4(w_2) = -1$, and there is no an element in $A(1,2)$ with such root.

If $\mathbf{q = 0}$ then $i_1 = n+2, 3 < i_2 < n+3$, and the action of $e_{33} - e_{n+2,n+2}$ implies $t + \sum_{i=1}^2 \delta_i = 2$. If $\delta_1 = 0$ then $g_{\epsilon_{n+2}-\epsilon_2} \otimes v \curvearrowright g_{\epsilon_{n+3}-\epsilon_1} \otimes v$ gives a $(e_{11} + e_{n+2,n+2})$-contradiction. Moreover, $p_{e_{22}+e_{44}}(w) = -1$ if $i_2 \neq 4$. If $n > 3$ then the action of $e_{22} + e_{55}$ gives a contradiction. Thus, we may assume $i_2 = 4, n = 3$ and

$$w = \phi(\alpha g_{\epsilon_5-\epsilon_2} \otimes v, \alpha^{\delta_2} g_{\epsilon_4-\epsilon_1} \otimes v, \underline{1 \otimes v}) \doteq g_{\epsilon_3-\epsilon_6}.$$

The action on $g_{\epsilon_6-\epsilon_1}$ gives

$$w_1 = \phi(\alpha g_{\epsilon_5-\epsilon_2} \otimes v, g_{\epsilon_6-\epsilon_1} \otimes v, \underline{1 \otimes v}) \neq 0.$$

We have $p_1(w_1) = -1 = p_4(w_1), p_2(w_1) = p_3(w_1) = 1, p_5(w_1) = 0$, i. e., $w_1 \doteq g_{\epsilon_2-\epsilon_4}$. From the action on $g_{\epsilon_4-\epsilon_2} \otimes v$, we arrive at

$$w_2 = \phi(g_{\epsilon_4-\epsilon_2} \otimes v, g_{\epsilon_6-\epsilon_1} \otimes v, \underline{1 \otimes v}) \neq 0.$$

We have $p_1(w_2) = p_2(w_2) = 0, p_3(w_2) = -1 = p_5(w_2), p_4(w_2) = 1$, and there is no an element in $A(1,3)$ with such root. ■

Lemma 4.4 *Let $V = V_\Lambda$ be an irreducible module over $A(1,n)$ with $\Lambda = (a_1, \ldots, a_{n+2})$, $a_2 = a \neq 0$ and $a_i = 0$ for $i \neq 2$. Suppose that $h = e_{11} + e_{33}$. Then it is impossible to have*

$$\phi(\overset{1+a}{u_1}, \ldots, \overset{1+a}{u_k}, \overset{a}{v_1}, \ldots, \overset{a}{v_{s-k}}) = g_{\epsilon_3-\epsilon_2}, \tag{6}$$

where the superscripts denote the h-weights, for some $k \geq 2$.

Proof. Denote by $\mathcal{E}_r$ the set of elements from $A(1,n)$ with h-weight equal to r . Suppose first that $s-k > 0$. From (6), acting on $1 \otimes v$, we have $w_1 = \phi(u_1, \ldots, u_k, 1 \otimes v, v_2, \ldots, v_{s-k}) \in \mathcal{E}_1 = \{g_{\epsilon_1-\epsilon_i}, g_{\epsilon_3-\epsilon_i} : i \neq 1, 3\}$. Assume that $1 \otimes v$ is odd. We may think that $w_1 \neq g_{\epsilon_3-\epsilon_2}$ since, otherwise, we can multiply w_1 by $g_{\epsilon_2-\epsilon_i}(i > 3)$. If $w_1 = g_{\epsilon_3-\epsilon_i}(i > 3)$ then, through the multiplication by $g_{\epsilon_1-\epsilon_3}$, we arrive at $w_1' = g_{\epsilon_1-\epsilon_i}$. Thus, $w_1 \doteq g_{\epsilon_1-\epsilon_i}$ and we may act on $t_1 = g_{\epsilon_j-\epsilon_1} \otimes v$ ($j = i$ if $i > 3$; $i \neq j > 3$ if $i = 2$), interchanging this element with u_1. We obtain $w_2 = \phi(\overset{a-1}{t_1}, \overset{a+1}{u_2}, \ldots, \overset{a+1}{u_k}, \overset{a}{v_1}, \ldots, \overset{a}{v_{s-k}}) \in \mathcal{E}_{-1} = \{g_{\epsilon_i-\epsilon_1}, g_{\epsilon_i-\epsilon_3} : i \neq 1, 3\}$, where, here and throughout this proof, the h-weights are above the elements. If $w_2 = g_{\epsilon_i-\epsilon_1}$ then we may multiply it by $g_{\epsilon_1-\epsilon_3}$ to arrive at $g_{\epsilon_i-\epsilon_3}$. So, $w_2 \doteq g_{\epsilon_i-\epsilon_3}$ and we replace u_2 by the action on $t_2 = g_{\epsilon_3-\epsilon_2} \overset{a+1}{\otimes} v$. Repeating this procedure, we substitute t_1 and arrive at a skewsymmetry contradiction. Suppose that $1 \otimes v$ is even. We have $w_1 = g_{\epsilon_3-\epsilon_i}$ ($i > 3$) or $w_1 = g_{\epsilon_1-\epsilon_j}$ ($j \neq 1, 3$). In the former case we may multiply w_1 by $g_{\epsilon_1-\epsilon_3}$ to get the latter one. In the latter case we can act on t_1 to arrive at w_2. Here we repeat the above argument to arrive at $w_3 = \phi(\underline{t_2}, v_1, \ldots, v_{s-k}) \in \mathcal{E}_1$. If $w_3 = g_{\epsilon_3-\epsilon_i}(i > 3)$ then we may multiply w_3 by $g_{\epsilon_i-\epsilon_{n+3}}$ to think that $w_3 = g_{\epsilon_3-\epsilon_{n+3}}$. We can do the same with $g_{\epsilon_1-\epsilon_i}(i > 3)$. Thus, we may assume that $\mathcal{E}_1 = \{g_{\epsilon_3-\epsilon_2}, g_{\epsilon_3-\epsilon_{n+3}}, g_{\epsilon_1-\epsilon_2}, g_{\epsilon_1-\epsilon_{n+3}}\}$. Notice that if $w_3 = g_{\epsilon_1-\epsilon_{n+3}}$ then we can multiply it by $g_{\epsilon_3-\epsilon_1}$ to arrive at $g_{\epsilon_3-\epsilon_{n+3}}$. Let $h_0 = e_{22} + e_{n+3,n+3}$. We remark that the element $g_{\epsilon_3-\epsilon_1}$ does not change either the h-weights or the h_0-weights. Therefore, we may replace all v_i with $g_{\epsilon_i-\epsilon_2} \otimes v, g_{\epsilon_{n+3}-\epsilon_2} \otimes v, 1 \otimes v$ (maybe multiplied by $g_{\epsilon_3-\epsilon_1}$). Adding the h_0-weights, we get $-2k - r = -2$ for some $r \in \mathbb{N}_0$, which is impossible because $k \geq 2$.

Now suppose that $s - k = 0$. Thus, (6) has the following shape

$$\phi(\overset{1+a}{u_1}, \ldots, \overset{1+a}{u_s}) = g_{\epsilon_3-\epsilon_2}. \tag{7}$$

We may multiply it by $g_{\epsilon_2-\epsilon_i}$ ($i > 3$) to assume that $\phi(u_1, \ldots, u_s) = g_{\epsilon_3-\epsilon_i}$. Through the multiplication by $g_{\epsilon_1-\epsilon_3}$, we may assume that $\phi(u_1, \ldots, u_s) = g_{\epsilon_1-\epsilon_i}$. We can now interchange u_1 and $g_{\epsilon_i-\epsilon_1} \overset{a-1}{\otimes} v$, and repeat the above described procedure to substitute

all u_i with $t_2 = g_{\epsilon_3-\epsilon_2} \otimes v$. Thus, we arrive at $\phi(\underline{t_2}) \in \mathcal{E}_1$. Considering the 3-weights, we obtain $\phi(t_2, t_2) = g_{\epsilon_3-\epsilon_4}$. From here, thinking in the 1-weights, we have a weight contradiction. ■

Lemma 4.5 *Let $V = V_\Lambda$ be an irreducible module over $G = A(1,n)$ with $\Lambda = (a_1, \ldots, a_{n+2})$. Suppose that $a_1 = 0, a_2 \neq 0$ and $\sum_{i=3}^{n+2} a_i = 1$. Assume that (G, V, ϕ) is a good triple. Then $0 < a_2 \leq 1/2$.*

Proof. Suppose that $\phi(u_1, \ldots, u_s) = g_{\epsilon_3-\epsilon_2}$ and consider $H = h_1 + h_2 = e_{11} + e_{33}$. By the nonzero action over $1 \otimes v$, we obtain $|1 - p_H(u_i) + a_2| \leq 1$. From here and taking into account that $\sum_{i=1}^{s} p_H(u_i) = 1$, we conclude that $a_2 \leq 1/2$. Let $H' = e_{22} + e_{n+3,n+3}$. Assume that $\phi(u_1, \ldots, u_s) = g_{\epsilon_{n+3}-\epsilon_2}$. By the action on $1 \otimes v$, we have $|-p_{H'}(u_i) + a_2 - 1| \leq 1$. Whence, $a_2 > 0$. ■

Theorem 4.1 *There are no simple finite-dimensional Filippov superalgebras of type $A(1,n)$ over Φ.*

Proof. Suppose that V is a finite-dimensional irreducible module over $G = A(1,n)$ with the highest weight $\Lambda = (a_1, \ldots, a_{2+n})$ $(a_1 \neq 0)$, and ϕ is a surjective skewsymmetric homomorphism from $\wedge^s V$ on G. Then there exist $u_i \in V_{\gamma_i}$ such that

$$\phi(u_1, \ldots, u_s) \;=\; g_{\epsilon_2-\epsilon_1}. \tag{8}$$

By Lemma 3.1, $\sum_{i=1}^{s} p_1(u_i) = -2$. From Lemma 3.2, $g_{\epsilon_2-\epsilon_1}^{a_1} \otimes v \neq 0$. Since ϕ is a skewsymmetric homomorphism, $\phi(u_1, \ldots, u_{i-1}, g_{\epsilon_2-\epsilon_1}^{a_1-1} \otimes v, u_{i+1}, \ldots, u_s) \neq 0$. As $p_1(g_{\epsilon_2-\epsilon_1}^{a_1-1} \otimes v) = 2 - a_1$, the inequality $|p_1(u_i) + a_1| \leq 2$ follows. From here we see that the required skewsymmetric homomorphism does not exist if $a_1 \geq 4$.

From now on, unless stated otherwise, we put the 1-weights above the elements.

Consider the case $\boxed{a_1 = 3}$. In this case, $p_1(u_i) < 0$. So, by (8), we have $\phi(\overset{-1}{u_1}, \overset{-1}{u_2}) = g_{\epsilon_2-\epsilon_1}$ and, acting on $1 \otimes v$, we arrive at $\phi(\overset{-1}{u_1}, 1 \overset{3}{\otimes} v) \doteq g_{\epsilon_1-\epsilon_2}$. Acting on $g_{\epsilon_2-\epsilon_1} \otimes v$, we obtain $\phi(g_{\epsilon_2-\epsilon_1} \overset{1}{\otimes} v, 1 \overset{3}{\otimes} v) \neq 0$, which is a weight contradiction.

Now let us take $\boxed{a_1 = 2}$. As $\sum_{i=1}^{s} p_1(u_i) = -2$ and, in this case, $p_1(u_i) \leq 0$, we can only have

$$\text{i) } \phi(\overset{-2}{u_1}, \overset{0}{u_2}, \ldots, \overset{0}{u_s}) = g_{\epsilon_2-\epsilon_1} \quad \text{or} \quad \text{ii) } \phi(\overset{-1}{u_1}, \overset{-1}{u_2}, \overset{0}{u_3}, \ldots, \overset{0}{u_s}) = g_{\epsilon_2-\epsilon_1}.$$

First consider i). Let us suppose that $1 \otimes v$ is even. Acting on $1 \otimes v$, we have $\phi(1 \overset{2}{\otimes} v, \overset{0}{u_2}, \ldots, \overset{0}{u_s}) \doteq g_{\epsilon_1-\epsilon_2}$. Then, acting twice on $g_{\epsilon_2-\epsilon_1} \otimes v$, we arrive at $\phi(\underline{g_{\epsilon_2-\epsilon_1} \otimes v}_2, u_3, \ldots, u_s) \neq 0$ which leads to a skewsymmetry contradiction. To finish the consideration of this subcase, suppose now that $1 \otimes v$ is odd. Then, acting on $1 \otimes v$ and, repeatedly, on $g_{\epsilon_2-\epsilon_1} \otimes v$, we get $\phi(1 \overset{2}{\otimes} v, \underline{g_{\epsilon_2-\epsilon_1} \overset{0}{\otimes} v}) \doteq g_{\epsilon_1-\epsilon_2}$. From here, analizing the 2-weights, we conclude that $a_2 = -1$. Assume that $\phi(u_1, \ldots, u_s) = g_{\epsilon_3-\epsilon_2}$. Consider

$h' = h_1 + h_2 = e_{11} + e_{33}$. From the nonzero action on $1 \otimes v$, we have $|-p_{h'}(u_i) + 2| \leq 1$. So, we obtain a contradiction because $\sum_{i=1}^{s} p_{h'}(u_i) = 1$. In the case ii), the multiplication by $g_{\epsilon_1-\epsilon_2}$ gives either

$$\phi(\overset{-1}{u_1}, \overset{-1}{u_2}, \overset{0}{u_3}, \ldots, \overset{2}{w}, \ldots, \overset{0}{u_s})(1 \otimes v) \neq 0 \quad \text{or} \quad \phi(\overset{-1}{u_1}, \overset{1}{v_2}, \overset{0}{u_3}, \ldots, \overset{0}{u_s})(1 \otimes v) \neq 0,$$

for some w, v_2. In both cases, replacing u_1 by $1 \otimes v$, we arrive at a weight contradiction.

Now take $\boxed{a_1 = 1}$. Consider (8) and $h'' = h_3 + \ldots + h_{2+n}$. By the nonzero action on $1 \otimes v$, we have $|-p_{h''}(u_i) + a_3 + \ldots + a_{2+n}| \leq 2$. So, we can deduce that $a_3 + \ldots + a_{2+n} < 3$ since otherwise it is impossible to have (8). As $a_r \in \mathbb{N}_0$ for $r \neq 2$ then

$$a_3 + \ldots + a_{2+n} \in \{0, 1, 2\}.$$

In what follows, we analise these three possibilities, numbered with I), II) and III), for $a_3 + \ldots + a_{2+n}$.

I) Assume that $a_3 + \ldots + a_{2+n} = 2$ and let $h = h_1 + h_3 + \ldots + h_{2+n}$. From (8), by the action on $1 \otimes v$, we arrive at $|1 - p_h(u_i)| \leq 2$. Thus, $p_h(u_i) \in \{-1, 0, 1, 2, 3\}$ and we obtain $\phi(\overset{-1}{u_1}, \overset{-1}{u_2}, \overset{0}{\overline{u_3, \ldots, u_s}}) = g_{\epsilon_2-\epsilon_1}$, where the h-weights are above the elements. The multiplication by $g_{\epsilon_1-\epsilon_2}$ leads to

$$\phi(\overset{-1}{u_1}, \overset{1}{w}, \overset{0}{\overline{u_3, \ldots, u_s}})(1 \overset{3}{\otimes} v) \neq 0 \text{ or } \phi(\overset{-1}{u_1}, \overset{-1}{u_2}, \overset{2}{\overline{\cdots}})(1 \overset{3}{\otimes} v) \neq 0,$$

for some w. In both cases, replacing u_1 by $1 \otimes v$, we get a weight contradiction.

II) Take h as above and suppose that $a_3 + \ldots + a_{2+n} = 1$. Once again by the action of $g_{\epsilon_2-\epsilon_1}$ on $1 \otimes v$, we get $|p_h(u_i)| \leq 2$. So, we have either $\phi(\overset{-1}{u_1}, \overset{-1}{u_2}, \overset{0}{\overline{u_3, \ldots, u_s}}) = g_{\epsilon_2-\epsilon_1}$ or $\phi(\overset{-2}{u_1}, \overset{0}{\overline{u_2, \ldots, u_s}}) = g_{\epsilon_2-\epsilon_1}$. In the former subcase, using the reasoning of I), we obtain more weight contradictions. In the latter subcase, multiplying by $g_{\epsilon_1-\epsilon_2}$, we arrive either at $\phi(\overset{-2}{u_1}, \overset{2}{\overline{\cdots}})(1 \overset{2}{\otimes} v) \neq 0$, which gives a weight contradiction, or at $\phi(\overset{0}{w}, \overset{0}{\overline{u_2, \ldots, u_s}})(1 \overset{2}{\otimes} v) \neq 0$. Thus, we have $\phi(1 \overset{2}{\otimes} v, \overset{0}{\overline{u_2, \ldots, u_s}}) \in \{g_{\epsilon_1-\epsilon_2}, g_{\epsilon_1-\epsilon_{n+3}}, g_{\epsilon_3-\epsilon_2}, g_{\epsilon_3-\epsilon_{n+3}}\}$. Taking into account that $p_h(g_{\epsilon_3-\epsilon_1}) = 0 = p_h(g_{\epsilon_{n+3}-\epsilon_2})$ and making adequate multiplications, we may assume that $\phi(\overset{2}{z}, \overset{0}{\overline{u_2, \ldots, u_s}}) \doteq g_{\epsilon_3-\epsilon_2}$, for some z. Suppose first that $g_{\epsilon_3-\epsilon_2} \otimes v \neq 0$. Then, by the action on $1 \overset{2}{\otimes} v$, we get a weight contradiction. Suppose now that $g_{\epsilon_3-\epsilon_2} \otimes v = 0$. Thus $a_2 = 0$. Assume that $\phi(u_1, \ldots, u_s) = g_{\epsilon_{n+3}-\epsilon_2}$. Let $h = e_{22} + e_{n+3,n+3}$ and, in what follows, consider the 2-weights above the elements and the h-weights underneath them. From the action on $1 \otimes v$, it is possible to conclude that $p_2(u_i), p_h(u_i) \in \{-2, -1, 0\}$. Thus we have $\phi(\underset{0}{\overset{-1}{u_1}}, \underset{0}{\overset{0}{u_2}}, \ldots, \underset{0}{\overset{0}{u_s}}) = g_{\epsilon_{n+3}-\epsilon_2}$. From here, considering the 2-weights and the h-weights, we arrive at

$$\phi(\underset{0}{\overset{-1}{u_1}}, 1 \underset{-1}{\overset{0}{\otimes}} v, \underset{0}{\overset{0}{u_3}}, \ldots, \underset{0}{\overset{0}{u_s}}) \doteq g_{\epsilon_i-\epsilon_2}, \text{ where } i \neq 2, 3, n+3.$$

If $g_{\epsilon_i-\epsilon_2}\otimes v\neq 0$ then the action on $1\underset{-1}{\otimes} v$ gives a weight contradiction. So, $g_{\epsilon_i-\epsilon_2}\otimes v=0$ and there exists $j>i$ such that $g_{\epsilon_i-\epsilon_2}g_{\epsilon_j-\epsilon_i}\otimes v\neq 0$. If $j\neq n+3$ then $p_h(g_{\epsilon_j-\epsilon_i}\otimes v)=-1$ and we obtain a weight contradiction through the action on $g_{\epsilon_j-\epsilon_i}\underset{-1}{\otimes} v$. Thus we may assume that $a_{n+2}=1$. Let us replace all u_k $(k\geq 3)$ with $g_{\epsilon_{n+3}-\epsilon_i}\otimes v$ and act one more time on such element. We have

$$\phi(1\overset{0}{\underset{-1}{\otimes}} v, g_{\epsilon_{n+3}-\epsilon_{i_1}}\overset{0}{\underset{0}{\otimes}} v,\ldots, g_{\epsilon_{n+3}-\epsilon_{i_{s-1}}}\overset{0}{\underset{0}{\otimes}} v)\neq 0.$$

Considering the h, 2 and 1-weights, we conclude that

$$\phi(1\overset{0}{\underset{-1}{\otimes}} v, g_{\epsilon_{n+3}-\epsilon_1}\overset{0}{\underset{0}{\otimes}} v,\ldots, g_{\epsilon_{n+3}-\epsilon_1}\overset{0}{\underset{0}{\otimes}} v)\doteq g_{\epsilon_1-\epsilon_{n+3}}. \tag{9}$$

From the multiplication by $g_{\epsilon_{n+3}-\epsilon_1}$, we have $\phi(\underline{g_{\epsilon_{n+3}-\epsilon_1}}\otimes v)\doteq h_1+h_2-h_3-\ldots-h_{n+2}$. The action on $1\otimes v$ leads to a contradiction.

III) We now have $a_t=0$ for $t\in\{3,\ldots,2+n\}$. Suppose that $a_2>0$. Consider $H'=h_1+h_2=e_{11}+e_{33}$ and assume that

$$\phi(u_1,\ldots,u_s)=g_{\epsilon_3-\epsilon_1}. \tag{10}$$

Acting on $1\otimes v$, we have $|-p_{H'}(u_i)+1+a_2|\leq 1$ and, consequently, $p_{H'}(u_i)\geq a_2>0$, being (10) impossible. Thus, $a_2\leq 0$.

Let us take first $a_2<0$. Suppose that

$$\phi(u_1,\ldots,u_s)=g_{\epsilon_3-\epsilon_2}. \tag{11}$$

By the action on $1\otimes v$, we arrive at $p_2(u_i)\leq 1+a_2$. We can't have (11) if $a_2<-1$. So, we conclude that $a_2\geq -1$. Now assume that $\phi(u_1,\ldots,u_s)=g_{\epsilon_{3+n}-\epsilon_2}$. Consider the action on $1\otimes v$. On one hand, we have $|-1-p_2(u_i)+a_2|\leq 1$ and, so, $a_2-2\leq p_2(u_i)\leq a_2<0$. On the other hand, $|2-p_1(u_i)|\leq 2$. Thus, we have $p_1(u_i)\in\{0,1\}$. From now on, in this subcase and unless stated otherwise, we will put the 1-weights above the elements and the 2-weights underneath them. Taking into account the 1-weights, we can only have

$$\phi(\overset{1}{u_1},\overset{0}{u_2},\ldots,\overset{0}{u_s})=g_{\epsilon_{3+n}-\epsilon_2}. \tag{12}$$

Acting on $1\otimes v$ allows us to obtain $\phi(\overset{1}{u_1}, 1\overset{1}{\underset{a_2}{\otimes}} v, \overset{0}{u_3},\ldots,\overset{0}{u_s})\doteq g_{\epsilon_1-\epsilon_2}$. Notice that a_2 has to be greater than -1; otherwise, we obtain a weight contradiction. By Lemma 4.1, $a_2\leq -\frac{1}{2}$. Therefore, we arrive at $a_2=-1/2$ and $\phi(\overset{1}{\underset{-1/2}{u_1}}, 1\overset{1}{\underset{-1/2}{\otimes}} v)\doteq g_{\epsilon_1-\epsilon_2}$. From the action on $g_{\epsilon_{3+n}-\epsilon_1}\otimes v$, we have

$$\phi(g_{\epsilon_{3+n}-\epsilon_1}\overset{0}{\underset{-1/2}{\otimes}} v, 1\overset{1}{\underset{-1/2}{\otimes}} v)\neq 0. \tag{13}$$

Taking into account the 1-weights, the 2-weights and the $(2+n)$-weights in (13), we conclude that we must have

$$\phi(g_{\epsilon_{n+3}-\epsilon_1} \otimes v, 1 \otimes v) \doteq g_{\epsilon_{n+3}-\epsilon_2}. \tag{14}$$

The nonzero action on $1 \otimes v$ leads to

$$\phi(1 \overset{1}{\underset{-1/2}{\otimes}} v, 1 \overset{1}{\underset{-1/2}{\otimes}} v) \doteq g_{\epsilon_1-\epsilon_2}. \tag{15}$$

If $1 \otimes v$ is even then we have a skewsymmetry contradiction. If $1 \otimes v$ is odd then we obtain the contradiction $0 = g_{\epsilon_1-\epsilon_3}$ from the multiplication by $g_{\epsilon_2-\epsilon_3}$ in (15).

Now let $a_2 = 0$. Observe that, by Lemma 4.2, all 2-weights of the elements of the pre-basis of V are zero or positive. Therefore, it is impossible to find $v_i \in V$ such that $\phi(v_1, \dots, v_s) = g_{\epsilon_1-\epsilon_2}$.

Now take $\boxed{a_1 = 0}$. Suppose first that $a_2 \neq 0$. Assume that $\phi(u_1, \dots, u_s) = g_{\epsilon_3-\epsilon_1}$ and $H'' = h_3 + \dots + h_{n+2} = e_{33} - e_{n+3,n+3}$. By the action on $1 \otimes v$, we have $|1 - p_{H''}(u_i) + a_3 + \dots + a_{n+2}| \leq 2$. From here, as $\sum_{i=1}^{s} p_{H''}(u_i) = 1$, we conclude that $a_3 + \dots + a_{n+2} < 2$. Taking into account (5), we have to study two subcases: 1) $a_3 = \dots = a_{n+2} = 0$; 2) $\sum_{i=3}^{n+2} a_i = 1$.

1) Consider $a_2 = a$, $h = e_{11} + e_{33}, v_i = g_{\epsilon_i-\epsilon_1}, w_j = g_{\epsilon_j-\epsilon_2}$ and

$$\phi(u_1, \dots, u_s) = g_{\epsilon_3-\epsilon_2}. \tag{16}$$

Then $\mathcal{E} = \langle v_{i_1} \dots v_{i_t} w_{j_1} \dots w_{j_r} \otimes v : i_p, j_q \neq 1, 2\rangle$ is a pre-basis of V. Note that $p_h(u_i) = a + t_i$, where $t_i \in \mathbb{Z}$ and $t_i \leq 1$. Then $\sum_{i=1}^{s}(t_i + a) = 1$ and, by the action of (16) on $1 \otimes v$, we have $|1 - (t_j + a) + a| \leq 1$ for all j. Thus, $t_j \in \{0, 1\}$ and $k(1 + a) + (s - k)a = 1$ for some $k \in \mathbb{N}_0$. By Lemma 4.3, $k \neq 0$. Notice also that $k \neq 1$. So, $k \geq 2$. We can also see that $-1 < a < 0$. From Lemma 4.4, we have that, for $k \geq 2$, this subcase can not occur.

2) Consider $\phi(u_1, \dots, u_s) = g_{\epsilon_3-\epsilon_2}$ and $H = e_{22} + e_{n+3,n+3}$. From the action on $1 \otimes v$, we obtain $| - 1 - p_H(u_i) + a_2 - 1| \leq 1$. From here and by Lemma 4.5, we get $-3 < p_H(u_i) \leq -\frac{1}{2}$. Therefore, $a_2 = \frac{1}{2}, s = 2$ and $p_H(u_1) = p_H(u_2) = -\frac{1}{2}$. Notice that, from the same action, for $h = e_{33} - e_{n+3,n+3}$, we get $p_h(u_i) \geq 0$. Henceforth, we have

$$\phi(\overset{-1/2}{\underset{1}{u_1}}, \overset{-1/2}{\underset{0}{u_2}}) = g_{\epsilon_3-\epsilon_2},$$

where the H-weights are above the elements and the h-weights underneath them. Acting on $1 \otimes v$, we obtain $\phi(\overset{-1/2}{\underset{1}{u_1}}, 1 \overset{-1/2}{\underset{1}{\otimes}} v) \doteq g_{\epsilon_3-\epsilon_{n+3}}$. The action on $g_{\epsilon_{n+3}-\epsilon_2} \otimes v$, taking into account the weights over $H, h, e_{11} + e_{n+3,n+3}$ and h_3, leads to $\phi(g_{\epsilon_{n+3}-\epsilon_2} \otimes v, 1 \otimes v) \doteq g_{\epsilon_1-\epsilon_{n+3}}$. Through the multiplication by $g_{\epsilon_{n+3}-\epsilon_2}$ we deduce that $1 \otimes v$ is even and we arrive at $\phi(\underline{g_{\epsilon_{n+3}-\epsilon_2} \otimes v}) \doteq g_{\epsilon_1-\epsilon_2}$. From here, multiplying by $g_{\epsilon_2-\epsilon_1}$, we have the contradiction $0 = h_1$.

At last, suppose that $a_2 = 0$. We may assume that $\sum_{i=3}^{n+2} a_i = a > 0$. Consider $\phi(u_1, \dots, u_s) = g_{\epsilon_{n+3}-\epsilon_2}$. The action on $1 \otimes v$ leads to $|-1-p_2(u_i)| \leq 1$ and, consequently, $p_2(u_i) \leq 0$ for all i. If $h = e_{22} + e_{n+3,n+3}$ then $|-p_h(u_i) - a| \leq 1$. So, $a = 1$ and $p_h(u_i) = 0$ for all i. Therefore, we have $\phi(\overset{-1}{\underset{0}{u_1}}, 1 \overset{0}{\underset{-1}{\otimes}} v, \overset{0}{\underset{0}{u_3}}, \dots, \overset{0}{\underset{0}{u_s}}) = g_{\epsilon_i-\epsilon_2}$ where $i \neq 2, 3, n+3$, the 2-weights and the h-weights are above and underneath the elements, respectively. If

$g_{\epsilon_i-\epsilon_2}\otimes v\neq 0$ then $u_1\mapsto 1\otimes v$ gives a h-weight contradiction. If $g_{\epsilon_i-\epsilon_2}\otimes v=0$ then there exists a $j>i$ such that $g_{\epsilon_i-\epsilon_2}g_{\epsilon_j-\epsilon_i}\otimes v\neq 0$. If $j\neq n+3$ then $p_h(g_{\epsilon_j-\epsilon_i}\otimes v)=-1$ and $u_1\mapsto g_{\epsilon_j-\epsilon_i}\otimes v$ gives a h-weight contradiction. Thus, we may assume that $a_{n+2}=1$. We can replace u_k with $g_{\epsilon_{n+3}-\epsilon_i}\otimes v$, $k\geq 3$. Continuing the process, we obtain $w_1=\phi(1\otimes v, g_{\epsilon_{n+3}-\epsilon_{i_1}}\otimes v,\ldots,g_{\epsilon_{n+3}-\epsilon_{i_{s-1}}}\otimes v)\neq 0$. Since the h_{n+2}-weight of w_1 is 1, $w_1\in\{g_{\epsilon_j-\epsilon_{n+3}}, g_{\epsilon_{n+2}-\epsilon_j} : j\neq n+2, n+3\}$. Let $h_0=e_{11}+e_{n+3,n+3}$. But $p_{h_0}(w_1)=-1$. Hence, either $w_1\doteq g_{\epsilon_{n+2}-\epsilon_1}$, which gives a h_1-weight contradiction, or $w_1\doteq g_{\epsilon_i-\epsilon_{n+3}}$. In the latter case, considering the h_k-weights for $k=1,2,3,\ldots$, we arrive at a weight contradiction.

To finish the proof, consider now $a_i=0$ for all i. Then V is trivial. ■

4.2 The main theorem

We can now state and prove the main result of this article.

Theorem 4.2 *There exist no simple finite-dimensional Filippov superalgebras of type $A(m,n)$ over Φ.*

Proof. We can suppose that $G=A(m,n)$ with $m\neq n$ and $m\geq 2$, because we have already proved that there exist no simple Filippov superalgebras of type $A(n,n)$ with $n\in\mathbb{N}$, [1], nor of type $A(1,n)$ with $n\in\mathbb{N}_0\setminus\{1\}$ and of type $A(0,n)$ with $n\in\mathbb{N}$, [2]. Assume that V is a finite-dimensional irreducible module over G with the highest weight $\Lambda=(a_1,\ldots,a_{m+n+1})$ $(a_1+\ldots+a_m\neq 0)$, and ϕ is a surjective skewsymmetric homomorphism from $\wedge^s V$ on G. Then there exist $u_i\in V_{\gamma_i}$ such that

$$\phi(u_1,\ldots,u_s) \;=\; g_{\epsilon_{m+1}-\epsilon_1}. \tag{17}$$

Let $H=h_1+\ldots+h_m=e_{11}-e_{m+1,m+1}$. By Lemma 3.1, $\sum_{i=1}^{s}p_H(u_i)=-2$. From Lemma 3.2, $g_{\epsilon_{m+1}-\epsilon_1}^{a_1+\ldots+a_m}\otimes v\neq 0$. Since ϕ is a skewsymmetric homomorphism, $\phi(u_1,\ldots,u_{i-1},g_{\epsilon_{m+1}-\epsilon_1}^{a_1+\ldots+a_m-1}\otimes v,u_{i+1},\ldots,u_s)\neq 0$. As $p_H(g_{\epsilon_{m+1}-\epsilon_1}^{a_1+\ldots+a_m-1}\otimes v)=2-a_1-\ldots-a_m$, the inequality $|p_H(u_i)+a_1+\ldots+a_m|\leq 2$ follows. From here we see that the required skewsymmetric homomorphism does not exist if $a_1+\ldots+a_m\geq 4$.

Throughout this proof, unless stated otherwise, we put the H-weights above the elements.

Consider the case $\boxed{a_1+\ldots+a_m=3}$. Then we have $\phi(\overset{-1}{u_1},\overset{-1}{u_2})=g_{\epsilon_{m+1}-\epsilon_1}$ and, acting on $1\otimes v$, we arrive at $\phi(\overset{-1}{u_1},1\overset{3}{\otimes}v)\doteq g_{\epsilon_1-\epsilon_{m+1}}$. By the action on $g_{\epsilon_{m+1}-\epsilon_1}\otimes v$, we obtain $\phi(g_{\epsilon_{m+1}-\epsilon_1}\overset{1}{\otimes}v,1\overset{3}{\otimes}v)\neq 0$, which is a weight contradiction.

Now let us take $\boxed{a_1+\ldots+a_m=2}$. Thus, there are two possibilities

$$\text{i) } \phi(\overset{-2}{u_1},\overset{0}{u_2},\ldots,\overset{0}{u_s})=g_{\epsilon_{m+1}-\epsilon_1} \quad\text{or}\quad \text{ii) } \phi(\overset{-1}{u_1},\overset{-1}{u_2},\overset{0}{u_3},\ldots,\overset{0}{u_s})=g_{\epsilon_{m+1}-\epsilon_1}.$$

First consider i). Let us suppose that $1\otimes v$ is even. Acting on $1\otimes v$, we have $\phi(1\overset{2}{\otimes}v,\overset{0}{u_2},\ldots,\overset{0}{u_s})\doteq g_{\epsilon_1-\epsilon_{m+1}}$. Then, acting twice on $g_{\epsilon_{m+1}-\epsilon_1}\otimes v$, we arrive at

$\phi(\underline{g_{\epsilon_{m+1}-\epsilon_1} \otimes v}_{2}, u_3, \ldots, u_s) \neq 0$ which leads to a skewsymmetry contradiction. To finish the consideration of this subcase, suppose now that $1 \otimes v$ is odd. Then, acting on $1 \otimes v$ and, repeatedly, on $g_{\epsilon_{m+1}-\epsilon_1} \otimes v$, we get $\phi(1 \overset{2}{\otimes} v, \underline{g_{\epsilon_{m+1}-\epsilon_1} \overset{0}{\otimes} v}) \doteq g_{\epsilon_1-\epsilon_{m+1}}$. From here, analyzing the $(m+1)$-weights, we conclude that $a_{m+1} = -1$. Assume that

$$\phi(u_1, \ldots, u_s) = g_{\epsilon_{m+2}-\epsilon_{m+1}}. \tag{18}$$

From the nonzero action on $1 \otimes v$, we arrive at $|3 - p_H(u_i)| \leq 2$. Consequently, we can't have (18). In the case ii), the multiplication by $g_{\epsilon_1-\epsilon_{m+1}}$ gives, for some v_i, either

$$\phi(\overset{-1}{u_1}, \overset{-1}{u_2}, \overset{0}{u_3}, \ldots, \overset{2}{v_i}, \ldots, \overset{0}{u_s})(1 \overset{2}{\otimes} v) \neq 0 \;\text{ or }\; \phi(\overset{-1}{u_1}, \overset{1}{v_2}, \overset{0}{u_3}, \ldots, \overset{0}{u_s})(1 \overset{2}{\otimes} v) \neq 0, \text{ for some } v_i, v_2.$$

In both cases, replacing u_1 by $1 \otimes v$, we arrive at a weight contradiction.

Now consider $\boxed{a_1 + \ldots + a_m = 1}$. Suppose that $a_{m+1} > 0$. Take $h = h_1 + \ldots + h_{m+1} = e_{11} + e_{m+2,m+2}$, and assume that

$$\phi(u_1, \ldots, u_s) = g_{\epsilon_{m+2}-\epsilon_1}. \tag{19}$$

Through the nonzero action on $1 \otimes v$, we have $p_h(u_i) > 0$ and (19) can not occur. Thus, $a_{m+1} \leq 0$.

I) Suppose that $a_{m+1} < 0$. Consider $\phi(u_1, \ldots, u_s) = g_{\epsilon_{m+2}-\epsilon_{m+1}}$. By the action on $1 \otimes v$, taking into account the $(m+1)$-weights and the h-weights, we arrive at $-1 \leq a_{m+1} \leq -\frac{1}{2}$.

Ia) Assume that $a_{m+1} \neq -1$. Let $\phi(u_1, \ldots, u_s) = g_{\epsilon_{m+2}-\epsilon_{m+1}}$. By the action on $1 \otimes v$, we have $p_H(u_i) \in \{0, 1, 2, 3, 4\}$. Consequently, after the action on $1 \otimes v$, we arrive at

$$\phi(\overset{1}{u_1}, 1 \overset{1}{\otimes} v, \overset{0}{u_3}, \ldots, \overset{0}{u_s}) \doteq g_{\epsilon_1-\epsilon_{m+1}}.$$

Replacing every $u_k (k \geq 3)$ by $g_{\epsilon_{m+2}-\epsilon_1} \otimes v$ and acting one more time on the mentioned element, we get $\phi(1 \overset{1}{\otimes} v, g_{\epsilon_{m+2}-\epsilon_1} \overset{0}{\otimes} v, \ldots, g_{\epsilon_{m+2}-\epsilon_1} \overset{0}{\otimes} v) \neq 0$. Analyzing the H, h and 1-weights involved, we conclude that $a_1 = 1$ and

$$\phi(1 \overset{1}{\otimes} v, g_{\epsilon_{m+2}-\epsilon_1} \overset{0}{\otimes} v, \ldots, g_{\epsilon_{m+2}-\epsilon_1} \overset{0}{\otimes} v) \doteq g_{\epsilon_1-\epsilon_i}, \; i \neq 1, 2, m+1, m+2.$$

Through the multiplication by $g_{\epsilon_{m+2}-\epsilon_1}$, we obtain $\phi(\underline{g_{\epsilon_{m+2}-\epsilon_1} \otimes v}) \doteq g_{\epsilon_{m+2}-\epsilon_i}, i \neq 1, 2, m+1, m+2$. Thus, considering the 2-weights, we have $i \neq 3$. Continuing the process, through the consecutive analises of the $2, 3, \ldots$-weights, we eliminate all the possibilities for i.

Ib) Assume that $a_{m+1} = -1$. 1) Consider $a_1 = 1$ and suppose that $\phi(u_1, \ldots, u_s) = g_{\epsilon_{m+2}-\epsilon_{m+1}}$. In this subcase, we put the $(m+1)$-weights above the elements and the 1-weights underneath them. Through the action on $1 \otimes v$, we have $p_{m+1}(u_i) \in \{-2, -1, 0\}, p_H(u_i) \in \{0, 1, 2, 3, 4\}$ and $p_1(u_i) \in \{-1, 0, 1, 2, 3\}$. If there is a k such that $p_1(u_k) = -1$ then the replacement of u_k by $1 \otimes v$ leads to a $(m+1), 1$-weights contradiction. Thus, $p_1(u_i) \geq 0$ and, through the action of $g_{\epsilon_{m+2}-\epsilon_{m+1}}$ on $1 \otimes v$, putting the H-weights in the third line, we arrive at

$$\phi(\underset{\substack{0\\1}}{\overset{0}{u_1}}, 1 \underset{\substack{1\\1}}{\overset{-1}{\otimes}} v, \underset{\substack{0\\0}}{\overset{0}{u_3}}, \ldots, \underset{\substack{0\\0}}{\overset{0}{u_s}}) \doteq g_{\epsilon_1-\epsilon_{m+1}}.$$

By the action on $g_{\epsilon_{m+1}-\epsilon_1} \otimes v$, we have $\phi(g_{\epsilon_{m+1}-\epsilon_1} \underset{\substack{0\\-1}}{\overset{0}{\otimes}} v, 1 \underset{\substack{1\\1}}{\overset{-1}{\otimes}} v, \underset{\substack{0\\0}}{\overset{0}{u_3}}, \ldots, \underset{\substack{0\\0}}{\overset{0}{u_s}}) \neq 0$. This is a weight contradiction since we don't have an element in $A(m,n)$ with the obtained $m+1, 1, H$-weights. 2) Now consider $a_m = 1$ and suppose that $\phi(u_1, \ldots, u_s) = g_{\epsilon_{m+2}-\epsilon_{m+1}}$. By the action on $1 \otimes v$, we conclude that $p_h(u_i) \in \{0,1,2\}$, $p_{m+1}(u_i) \in \{-2,-1,0\}$ and $p_m(u_i) \in \{0,1,2,3,4\}$. So, we have

$$\phi(\underset{0}{\overset{1}{u_1}}, \underset{0}{\overset{0}{u_2}}, \ldots, \underset{0}{\overset{0}{u_s}}) = g_{\epsilon_{m+2}-\epsilon_{m+1}}, \tag{20}$$

where the h-weights are above the elements and the $(m+1)$-weights are underneath them. From the action on $1 \otimes v$, we obtain

$$\phi(1 \underset{-1}{\overset{0}{\otimes}} v, \underset{0}{\overset{0}{u_2}}, \ldots, \underset{0}{\overset{0}{u_s}}) \doteq g_{\epsilon_1-\epsilon_{m+2}} (g_{\epsilon_i-\epsilon_{m+1}}, i \neq 1, m+1, m+2).$$

Consider the former possibility. As $p_m(g_{\epsilon_{m+2}-\epsilon_{m+1}}) = 1$, $p_m(g_{\epsilon_1-\epsilon_{m+2}}) = 0$, $p_m(1 \otimes v) = 1$ then, in (20), $p_m(u_1) = 2$ and the sum of the m-weights of the remaining elements is equal to -1, which is impossible. Consider the latter occasion. Notice that $p_m(g_{\epsilon_i-\epsilon_{m+1}})$ is either 1 (when $i \neq m$) or 2 (when $i = m$). If $i \neq m$ then $p_m(u_1) = 1$ and the change $u_2 \mapsto 1 \otimes v$ in (20) gives a $h, m+1, m$-weights contradiction. If $i = m$ then $p_m(u_1) = 0$ and, acting on $1 \otimes v$ in (20), we have $\phi(1 \underset{\substack{-1\\1}}{\overset{0}{\otimes}} v, \underbrace{\underset{0}{\overset{0}{u_2}}, \ldots, \underset{0}{\overset{0}{u_s}}}_{1}) \doteq g_{\epsilon_m-\epsilon_{m+1}}$, where the m-weights of the elements are in the third weight line. Through the action on $g_{\epsilon_{m+1}-\epsilon_m} \otimes v$, we obtain $\phi(1 \underset{\substack{-1\\1}}{\overset{0}{\otimes}} v, g_{\epsilon_{m+1}-\epsilon_m} \underset{\substack{0\\-1}}{\overset{0}{\otimes}} v, \underset{\substack{0\\0}}{\overset{0}{u_3}}, \ldots, \underset{\substack{0\\0}}{\overset{0}{u_s}}) \neq 0$, one more weight contradiction. 3) Assume that there exists a $j \in \{2, \ldots, m-1\}$ such that $a_j = 1$. In this subcase, we put the $(m+1)$-weights above the elements and the j-weights underneath them. Assume that $\phi(u_1, \ldots, u_s) = g_{\epsilon_{j+1}-\epsilon_j}$. The action on $1 \otimes v$ allows us to conclude that $p_{m+1}(u_i) \in \{-2,-1,0\}$ and $p_j(u_i) \in \{-3,-2,-1,0,1\}$. If, for example, $p_j(u_1) = 1$ then, through the mentioned action, we obtain the weight contradiction $\phi(1 \underset{1}{\overset{-1}{\otimes}} v, \underbrace{\overset{0}{u_2}, \ldots, \overset{0}{u_s}}_{-3}) \neq 0$. Suppose now that $\phi(\underset{0}{\overset{0}{u_1}}, \ldots, \underset{0}{\overset{0}{u_{s-2}}}, \underset{-1}{\overset{0}{u_{s-1}}}, \underset{-1}{\overset{0}{u_s}}) = g_{\epsilon_{j+1}-\epsilon_j}$. Multiplying the last equality by $g_{\epsilon_j-\epsilon_{j+1}}$ and acting on $1 \otimes v$, we get $\phi(\underbrace{\overset{0}{u_1}, \ldots, \overset{0}{u_{s-1}}}_{1}, 1 \underset{1}{\overset{-1}{\otimes}} v) \neq 0$, which is a weight contradiction. Finally, we study the subcase $u = \phi(\underset{0}{\overset{0}{u_1}}, \ldots, \underset{0}{\overset{0}{u_{s-1}}}, \underset{-2}{\overset{0}{u_s}}) = g_{\epsilon_{j+1}-\epsilon_j} (*)$. The action on $1 \otimes v$ leads to $w = \phi(\underset{0}{\overset{0}{u_1}}, \ldots, \underset{0}{\overset{0}{u_{s-1}}}, 1 \underset{1}{\overset{-1}{\otimes}} v) \in \{g_{\epsilon_j-\epsilon_{m+1}}, g_{\epsilon_j-\epsilon_{m+2}}\}$. As $p_{j-1}(u) = 1$ and $p_{j-1}(w) = -1$ then $p_{j-1}(u_s) = 2$. So, through the action on $1 \otimes v$ in $(*)$, we obtain the weight contradiction $\phi(1 \underset{\substack{1\\0}}{\overset{-1}{\otimes}} v, \underbrace{\underset{0}{\overset{0}{u_2}}, \ldots, \underset{0}{\overset{0}{u_{s-1}}}}_{\geq 0}, \underset{\substack{-2\\2}}{\overset{0}{u_s}}) \neq 0$, where the third weight line refers to h_{j-1}.

II) Consider the case $a_{m+1} = 0$. Assume that $\phi(u_1, \ldots, u_s) = g_{\epsilon_{m+2}-\epsilon_1}$. Take h as above. Through the action on $1 \otimes v$, we have $p_{m+1}(u_i), p_h(u_i) \in \{0, 1, 2\}$. Thus

$$\phi(\overset{1}{\underset{0}{u_1}}, \overset{0}{\underset{0}{u_2}}, \ldots, \overset{0}{\underset{0}{u_s}}) = g_{\epsilon_{m+2}-\epsilon_1}, \tag{21}$$

where, here and in what follows, we consider the $(m+1)$-weights above the elements and the h-weights underneath them. By the action on $1 \otimes v$ in (21), we arrive at

$$\phi(\overset{1}{\underset{0}{u_1}}, 1 \overset{0}{\underset{1}{\otimes}} v, \overset{0}{\underset{0}{u_3}}, \ldots, \overset{0}{\underset{0}{u_s}}) = g_{\epsilon_{m+2}-\epsilon_i}, \text{with } i \neq 1, m+1, m+2. \tag{22}$$

If $g_{\epsilon_{m+2}-\epsilon_i} \otimes v \neq 0$ then we get a weight contradiction from the action on $1 \underset{1}{\otimes} v$. So, $g_{\epsilon_{m+2}-\epsilon_i} \otimes v = 0$ and there exists a $j < i$ such that $g_{\epsilon_{m+2}-\epsilon_i} g_{\epsilon_i-\epsilon_j} \otimes v \neq 0$. If $j \neq 1$ then $p_h(g_{\epsilon_i-\epsilon_j} \otimes v) = 1$ and from the action on $g_{\epsilon_i-\epsilon_j} \otimes v$ arises a weight contradiction. We may assume that $a_1 = 1$. Let us replace all u_k in (22), for $k \geq 3$, by $g_{\epsilon_{i_k}-\epsilon_1} \overset{0}{\underset{0}{\otimes}} v$ and act one more time on $g_{\epsilon_{i_1}-\epsilon_1} \overset{0}{\underset{0}{\otimes}} v$. Then, looking at the $(m+1), h, 1$-weights, we obtain

$$u := \phi(1 \otimes v, g_{\epsilon_{i_1}-\epsilon_1} \otimes v, g_{\epsilon_{i_3}-\epsilon_1} \otimes v, \ldots, g_{\epsilon_{i_s}-\epsilon_1} \otimes v) = g_{\epsilon_1-\epsilon_t}(g_{\epsilon_{m+2}-\epsilon_{m+1}}),$$

with $t \neq 1, 2, m+1, m+2$. Note that $i_k < m+1$, since otherwise $q := i_k = m+2$ for some k and we arrive at $(e_{11} + e_{qq})$-contradiction. Moreover, if $u = g_{\epsilon_{m+2}-\epsilon_{m+1}}$ then the multiplication on $g_{\epsilon_1-\epsilon_{m+2}}$ gives a contradiction. Now, for $t = m, m-1, \ldots, 2$, considering, consecutively, all these t-weights, we arrive at a weight contradiction.

Finally, suppose that $\boxed{a_1 + \ldots + a_m = 0}$. As $A(m,n) \simeq A(n,m)$, [7, Section 4.2.2], then $a_{m+2} + \ldots + a_{m+n+1} = 0$. Thus, consider $a_t = 0$ for $t \neq m+1$ and $a_{m+1} = a \neq 0$. Assume that $h = e_{11} + e_{m+n+2,m+n+2}$. Let $w = \phi(u_1, \ldots, u_s) = g_{\epsilon_{m+1}-\epsilon_2}$. Then $p_h(w) = 0$ and $p_h(u_i) = a + \epsilon_i$ with $\epsilon_i \in \mathbb{Z}$. Take $x = g_{\epsilon_{m+n+2}-\epsilon_{m+1}} \otimes v$. We have $wx \neq 0$ and $p_h(x) = a + 1$. If $p_h(u_i) = a - \epsilon_i$, for some i and $\epsilon_i \in \mathbb{N}$, then $u_i \curvearrowright x$ gives a weight contradiction. Therefore, $a < 0$. Now let $w = \phi(u_1, \ldots, u_s) = g_{\epsilon_{m+3}-\epsilon_{m+2}}$. Then $p_h(w) = 0$ and $p_h(u_i) = a + \epsilon_i$ with $\epsilon_i \in \mathbb{Z}$. Take $x = g_{\epsilon_{m+2}-\epsilon_1} \otimes v$. Then $p_h(x) = a - 1$ and $wx \neq 0$. If $p_h(u_j) = a + \epsilon_j$, for some j and $\epsilon_j \in \mathbb{N}$, then $u_j \curvearrowright x$ gives a weight contradiction. Henceforth, $a > 0$. Thus, $a = 0$ and the module is trivial. This finishes the proof of the theorem. ■

Corollary 4.1 *There is no simple finite-dimensional Filippov superalgebra $\mathcal{F}$ of type $A(m,n)$ such that $\mathcal{F}$ is a highest weight module over $A(m,n)$.*

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